

The Poset Metrics That Allow Binary Codes of Codimension m to be m -, $(m-1)$ -, or $(m-2)$ -Perfect

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Abstract—A binary poset code of codimension m (of cardinality 2^{n-m} , where n is the code length) can correct maximum m errors. All possible poset metrics that allow codes of codimension m to be m -, $(m-1)$ -, or $(m-2)$ -perfect are described. Some general conditions on a poset which guarantee the nonexistence of perfect poset codes are derived; as examples, we prove the nonexistence of r -perfect poset codes for some r in the case of the crown poset and in the case of the union of disjoint chains.

Index terms—perfect codes, poset codes

I. INTRODUCTION

We study the problem of existence of perfect codes in poset metric spaces, which are a generalization of the Hamming metric space, see [2]. There are several papers [1], [3], [4] on the existence of 1-, 2-, or 3-error-correcting poset codes. The approach of the present work is opposite; we start to classify posets that admit the existence of perfect codes correcting as many as possible errors with respect to the code length and dimension, i.e., when the number of errors is close to the code codimension.

As stated by Lemma 2-5 below, the codimension m of an r -error-correcting $(n, 2^{n-m})$ code cannot be less than r . And the posets that allow binary poset-codes of codimension m to be m -perfect have a simple characterization (Theorem 2-6).

The main results of this work, stated by Theorem 4-4 and Theorem 6-1, are criteria for the existence of $(m-1)$ - and $(m-2)$ -perfect $(n, 2^{n-m})$ P -codes. The intermediate results formulated as lemmas may also be useful for the description of other poset structures admitting perfect poset codes.

Let $P = ([n], \preceq)$ be a poset, where $[n] \triangleq \{1, \dots, n\}$. A subset I of $[n]$ is called an *ideal*, or *downset* (an *upset*, or *filter*) iff for each $a \in I$ the relation $b \preceq a$ (respectively, $b \succeq a$) means $b \in I$. For $a_1, \dots, a_i \in P$ denote by $\langle a_1, \dots, a_i \rangle$ or $\langle \{a_1, \dots, a_i\} \rangle$ the *principal ideal* of $\{a_1, \dots, a_i\}$, i.e., the minimal ideal that contains a_1, \dots, a_i ; and by $\succ a_1, \dots, a_i <$ or $\succ \{a_1, \dots, a_i\} <$, the minimal upset that contains a_1, \dots, a_i .

Denote by $\mathcal{I}_P^r \subset 2^{[n]}$ the set of all r -ideals (i.e., ideals of cardinality r) of P , where $r \in \{0, 1, \dots, n\}$.

If S is an arbitrary set (poset), then the set of all subsets of S is denoted by 2^S . The set $2^{[n]}$ will be also denoted as F^n , and we will not distinguish subsets of $[n]$ from their characteristic vectors; for example, $2^{[5]} \ni \{2, 4, 5\} = (01011) \in F^5$.

If $\bar{x} \in 2^{[n]}$, then the P -weight $w_P(\bar{x})$ of \bar{x} is the cardinality of $\langle \bar{x} \rangle$. Now, for two elements $\bar{x}, \bar{y} \in F^n$ we can define the P -distance

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$d_P(\bar{x}, \bar{y}) \triangleq w_P(\bar{x} + \bar{y})$, where $+$ means the symmetrical difference in terms of subsets of $[n]$ and the mod 2 addition in terms of their characteristic functions.

For $r \in \{0, \dots, n\}$ we denote by $\mathcal{B}_P^r \triangleq \{\bar{x} \in F^n \mid w_P(\bar{x}) \leq r\}$ the *ball* of radius r with center in the all-zero vector $\bar{0}$. A subset \mathcal{C} of F^n is called an r -error-correcting P -code (r -perfect P -code) iff each element \bar{x} of F^n has at most one (respectively, exactly one) representation in the form $\bar{x} = \bar{c} + \bar{b}$, where $\bar{c} \in \mathcal{C}$ and $\bar{b} \in \mathcal{B}_P^r$. In other words, the balls of radius r centered in the codewords of an r -error-correcting P -code \mathcal{C} are mutually disjoint (the *ball-packing condition*) and, if \mathcal{C} is r -perfect, cover all the space F^n . As a consequence,

$$|\mathcal{C}| \leq |F^n|/|\mathcal{B}_P^r|$$

(the *ball-packing bound*), where equality is equivalent to the r -perfectness of \mathcal{C} .

For the rest of the paper we will use the following notations. Let $\mathcal{C} \subset F^n$ be a P -code and $\bar{0} \in \mathcal{C}$; denote

- $m \triangleq n - \log_2 |\mathcal{C}|$,
- $P^r \triangleq \bigcup_{I \in \mathcal{I}_P^r} I \subseteq [n]$,
- $u \triangleq |\bigcap_{I \in \mathcal{I}_P^r} I|$,
- $\tilde{P}^r \triangleq P^r \setminus \bigcap_{I \in \mathcal{I}_P^r} I$ (studying r -perfect codes, we can call \tilde{P}^r the “essential part” of P ; indeed, the ball \mathcal{B}_P^r is the Cartesian product of $\mathcal{B}_{\tilde{P}^r}^{r-u}$ and $2^{P^r \setminus \tilde{P}^r}$),
- $\lambda \triangleq |P^r| - r$,
- $\max(R)$ denotes the set of maximal elements of a poset R ,
- $\min(R)$ denotes the set of minimal elements of a poset R ,
- $k \triangleq |\max(\tilde{P}^r)|$.

Note that u , λ , and k depend on P and r though the notations do not reflect this dependence explicitly.

II. m -ERROR-CORRECTING POSET CODES

We start with several auxiliary statements. The first one is easy and well known.

Proposition 2-1:

- a) Let $0 \leq r \leq r' \leq n$ and $I \in \mathcal{I}_P^r$; then there exists $I' \in \mathcal{I}_P^{r'}$ such that $I \subseteq I'$.
- b) Let $0 \leq r' \leq r \leq n$ and $I \in \mathcal{I}_P^r$; then there exists $I' \in \mathcal{I}_P^{r'}$ such that $I' \subseteq I$.

Proof: If $r' = r$, then $I' = I$ in both cases.

a) In the case $r' = r + 1$ let j be a minimal element of $P \setminus I$; then $I' \triangleq I \cup \{j\}$ satisfies the condition.

b) In the case $r' = r - 1$ let j be a maximal element of I ; then $I' \triangleq I \setminus \{j\}$ satisfies the condition.

The general cases $r' = r \pm t$ are proved by induction. ■

Corollary 2-2: For each r' from 0 to n the set $\mathcal{I}_P^{r'}$ is not empty.

Proof: Assigning $r = 0$ and $I = \emptyset$ in Proposition 2-1 we get at least one ideal in \mathcal{I}_P^0 . ■

Proposition 2-3: $\mathcal{B}_P^r = \bigcup_{I \in \mathcal{I}_P^r} 2^I$.

Proof: Let $\bar{x} \in \mathcal{B}_P^r$, i.e., $w_P(\bar{x}) = |\langle \bar{x} \rangle| \leq r$. By Proposition 2-1 there exists an ideal $I \in \mathcal{I}_P^r$ such that $\langle \bar{x} \rangle \subseteq I$. So, we have $\bar{x} \subseteq \langle \bar{x} \rangle \subseteq I$, and $\bar{x} \in 2^I$.

Conversely, if $\bar{x} \in 2^I$ for some $I \in \mathcal{I}_P^r$, then $\langle \bar{x} \rangle \subseteq I$ and $w_P(\bar{x}) = |\langle \bar{x} \rangle| \leq |I| = r$. ■

Since $|\mathcal{I}_P^r| \geq 1$ by Corollary 2-2, we immediately obtain

Corollary 2-4: $|\mathcal{B}_P^r| \geq 2^r$.

The following lemma is straightforward from the ball-packing bound and Corollary 2-4.

Lemma 2-5: If a $(n, 2^{n-m})$ P -code $\mathcal{C} \subset F^n$ is r -error-correcting, then $r \leq m$.

Theorem 2-6 (characterization of m -error-correcting P -codes):

An $(n, 2^{n-m})$ code \mathcal{C} is an m -error-correcting P -code if and only if the following two conditions hold:

- a) \mathcal{I}_P^m contains exactly one ideal I ;
- b) there is a function $f: 2^{P \setminus I} \rightarrow 2^I$ such that $\mathcal{C} = \{Y \cup f(Y) \mid Y \in 2^{P \setminus I}\}$, i.e., the code \mathcal{C} is systematic with information symbols $P \setminus I$ and check symbols I .

Every m -error-correcting P -code is an m -perfect P -code.

Proof: We first show that a) and b) hold for any m -error-correcting P -code \mathcal{C} . If \mathcal{I}_P^m contains more than one ideal, then $|\mathcal{B}_P^m| > 2^m$, and we have a contradiction with the ball-packing condition. So, \mathcal{I}_P^m contains exactly one ideal, say, I .

If there is no such a function as in b), then there are two codewords $\bar{c}_1, \bar{c}_2 \in \mathcal{C}$ that coincide in $P \setminus I$. Then $\bar{c}_1 + \bar{c}_2 \subseteq I$, and $d_P(\bar{c}_1, \bar{c}_2) = |\langle \bar{c}_1 + \bar{c}_2 \rangle| \leq |I| = m$; therefore \mathcal{C} is not m -error-correcting. So, b) is a necessary condition.

Assume a) and b) hold. We show that \mathcal{C} is an m -perfect code. We need to check that for each $\bar{y} \in F^n$ there exists a unique $\bar{c} \in \mathcal{C}$ such that $d_P(\bar{c}, \bar{y}) \leq m$. Such \bar{c} can be defined by $\bar{c} = \bar{y} \cap (P \setminus I) \cup f(\bar{y} \cap (P \setminus I))$. It is a code vector by the definition of f ; and $d_P(\bar{c}, \bar{y}) \leq m$ because $\bar{c} + \bar{y} \subseteq I$. The uniqueness follows from the equalities $|\mathcal{C}| = 2^{n-m} = |F^n|/|\mathcal{B}_P^m|$. ■

III. USEFUL STATEMENTS

Proposition 3-1: A P -code \mathcal{C} is r -error-correcting if and only if for each different $\bar{c}_1, \bar{c}_2 \in \mathcal{C}$ and each $I', I'' \in \mathcal{I}_P^r$ we have $\bar{c}_1 + \bar{c}_2 \not\subseteq I' \cup I''$.

Proof: Only if: Assume that there exist $\bar{c}_1, \bar{c}_2 \in \mathcal{C}$ and $I', I'' \in \mathcal{I}_P^r$ such that $\bar{c}_1 + \bar{c}_2 \subseteq I' \cup I''$. Consider the vector $\bar{v} \triangleq \bar{c}_1 + (\bar{c}_1 + \bar{c}_2) \cap I'$. We have that $d_P(\bar{v}, \bar{c}_1) = w_P(\bar{v} + \bar{c}_1) = w_P((\bar{c}_1 + \bar{c}_2) \cap I') \leq |I'| = r$. On the other hand, $d_P(\bar{v}, \bar{c}_2) = w_P(\bar{v} + \bar{c}_2) = w_P(\bar{c}_1 + \bar{c}_2 + (\bar{c}_1 + \bar{c}_2) \cap I') = w_P((\bar{c}_1 + \bar{c}_2) \setminus I') \leq |I''| = r$ because $(\bar{c}_1 + \bar{c}_2) \setminus I' \subseteq I''$ by assumption. So, \mathcal{C} is not r -error-correcting.

If: Let the P -code \mathcal{C} be not r -error-correcting. Then there exist two different codewords $\bar{c}_1, \bar{c}_2 \in \mathcal{C}$ and a vector $\bar{v} \in F^n$ such that $d_P(\bar{v}, \bar{c}_1) = |\langle \bar{v} + \bar{c}_1 \rangle| \leq r$ and $d_P(\bar{v}, \bar{c}_2) = |\langle \bar{v} + \bar{c}_2 \rangle| \leq r$. By Proposition 2-3 we have that $\bar{v} + \bar{c}_1 \subseteq I'$ and $\bar{v} + \bar{c}_2 \subseteq I''$ for some $I', I'' \in \mathcal{I}_P^r$. Then $\bar{c}_1 + \bar{c}_2 = (\bar{v} + \bar{c}_1) + (\bar{v} + \bar{c}_2) \subseteq I' \cup I''$. ■

The statement (Corollary 3-9) that we will use for proving the main result can be derived from each of the following two lemmas.

Lemma 3-2: If there is an r -error-correcting $(n, 2^{n-m})$ P -code, then $|I' \cup I''| \leq m$ for each $I', I'' \in \mathcal{I}_P^r$.

Proof: Assume $|I' \cup I''| > m$, i.e., $|P \setminus (I' \cup I'')| < n - m$. Since $|\mathcal{C}| = 2^{n-m}$, there are two different codewords $\bar{c}_1, \bar{c}_2 \in \mathcal{C}$ that coincide in $P \setminus (I' \cup I'')$. This contradicts Proposition 3-1. ■

Lemma 3-3: Suppose there exists a vector $\bar{v} \in F^n \setminus \mathcal{B}_P^r$ such that for each $I \in \mathcal{I}_P^r$ it is true that $\bar{v} \cup I \subseteq I' \cup I''$ for some $I', I'' \in \mathcal{I}_P^r$. Then no r -perfect P -codes exist.

Proof: Assume the contrary, i.e., there exists an r -perfect P -code \mathcal{C} and $\bar{0} \in \mathcal{C}$. Let \bar{c} be a codeword such that $d_P(\bar{v}, \bar{c}) \leq r$. Then $\bar{v} + \bar{c} \in \mathcal{B}_P^r$ and by Proposition 2-3 it is true that $\bar{v} + \bar{c} \subseteq I$ for some $I \in \mathcal{I}_P^r$. Therefore $\bar{c} \subseteq \bar{v} \cup I$. By hypothesis, $\bar{c} \subseteq I' \cup I''$ for some $I', I'' \in \mathcal{I}_P^r$, and we get a contradiction with Proposition 3-1. ■

The following two corollaries are weaker than Lemma 3-3, but their conditions are more handy for verification. Given an ideal V , denote

$$W(V) \triangleq [n] \setminus \langle \max(V) \rangle.$$

It is clear that $W(V)$ is an ideal and it includes $V \setminus \max(V)$.

Corollary 3-4: Suppose $V \in \mathcal{I}_P^{r+1}$. Then the following conditions are equivalent and imply the nonexistence of r -perfect P -codes:

- a) every $I \in \mathcal{I}_P^r$ contains at least one element b of $\max(V)$;
- b) $|W(V)| < r$.

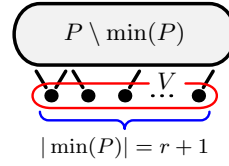


Fig. 1. Example 3-5

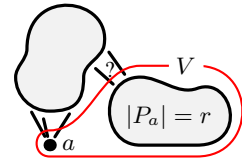


Fig. 2. Example 3-6

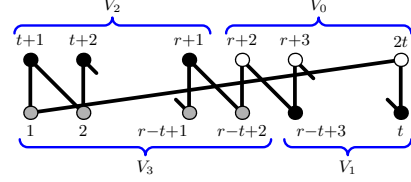


Fig. 3. Example 3-8: the crown poset

Proof: Assume a) does not hold, i.e., there is $I \in \mathcal{I}_P^r$ such that $I \cap \max(V) = \emptyset$. Then $I \subseteq W(V)$ and $|W(V)| \geq |I| \geq r$. So, b) implies a).

Assume b) does not hold, i.e., $|W(V)| \geq r$. By Proposition 2-1 there exists an ideal $I \subseteq W(V)$ of cardinality r . Then $I \cap \max(V) = \emptyset$ and a) does not hold too. So, a) implies b).

Assume a) holds. By Lemma 3-3 with $\bar{v} = V$, $I' = V \setminus \{b\}$, $I'' = I$, we get the nonexistence of r -perfect P -codes. ■

Example 3-5: (Fig. 1.) If $|\min(P)| = r + 1$, then the ideal $V \triangleq \min(P)$ satisfies the conditions of Corollary 3-4; hence, there exist no r -perfect P -codes.

Example 3-6: (Fig. 2.) Let a be a minimal element of P and let P_a be the ideal $P_a \triangleq \{b \mid b \not\leq a\}$. If $|P_a| = r$, then no r -perfect P -codes exist, because the ideal $V \triangleq \{a\} \cup P_a$ satisfies the conditions of Corollary 3-4.

Example 3-7: Let P be a poset that consists of $t \geq 2$ disjoint chains and $t - 1 \leq r < n$. Then no r -perfect P -codes exist. Indeed, it is easy to see that an arbitrary $(r + 1)$ -ideal V that contains all t minimal elements of P satisfies the conditions of Corollary 3-4.

The subcase of Example 3-7 where $t = 2$ and the chains are equipotent coincides with the binary case of [2, Theorem 2.2] (which was proved for codes over arbitrary finite field).

In the next example we see that r -perfect P -codes do not exist for sufficiently large r if P is the crown, i.e., $n = 2t \geq 4$, $i \leq t + i$, $i + 1 \leq t + i$, $1 \leq 2t$, $t \leq 2t$, and these are the only strict comparabilities in P . The existence of 1-, 2-, and 3-perfect crown-codes has been studied in [1], [4].

Example 3-8: Let P be a crown with $n = 2t \geq 6$ and let $t/2 \leq r < n$; then no r -perfect P -codes exist unless $t = 3$ and $r = 4$. Indeed, it is not difficult to check that condition b) of Corollary 3-4 is satisfied with the following choice of V : if $t/2 \leq r < t$ then $V = [t] \setminus \{2, 4, \dots, 2(t - r - 1)\}$; if $t \leq r < n$ then $V = [r + 1]$ (Fig. 3, where $V = V_1 + V_2 + V_3$).

Corollary 3-9: Suppose there exist two different ideals $I', I'' \in \mathcal{I}_P^r$ such that $P^r = I' \cup I''$. Then no r -perfect P -codes exist.

Proof: Approach 1: apply Lemma 3-3 with $\bar{v} = P^r$. Approach 2: if r -perfect P -codes exist, then $|P^r| > m$ (indeed, 2^{P^r} includes the ball \mathcal{B}_P^r of cardinality 2^m , but at least one point, P^r , of 2^{P^r} does not belong to \mathcal{B}_P^r) and we have a contradiction with Lemma 3-2. ■

Let \mathcal{W}_n^r be the set of all r -subsets of $[n]$. Define the distance $d_J(I, I') \triangleq |I \setminus I'|/2$ (the Johnson distance) between any I and I' from \mathcal{W}_n^r . Let G_n^r be the adjacency graph of \mathcal{W}_n^r , where two subsets $I, I' \in \mathcal{W}_n^r$ are adjacent iff $d_J(I, I') = 1$; and let $G_n^r(P)$ be the subgraph of G_n^r induced by \mathcal{I}_P^r . The following known fact can be easily proved by induction.

Proposition 3-10: Let I and I' be ideals from \mathcal{I}_P^r . Then I and I' are connected by a path of length $d_J(I, I')$ in the graph $G_n^r(P)$.

Proof: If $d_J(I, I') = 0$ or 1 , it is trivial.

Assume the statement holds for $d_J(I, I') = \delta - 1 \geq 1$.

Let $d_J(I, I') = \delta$. Let v be a minimal element of $I \setminus I'$ and v' be a maximal element of $I' \setminus I$. It is not difficult to check that the set $I'' \triangleq \{v\} \cup I' \setminus \{v'\}$ is an ideal from \mathcal{I}_P^r . Since $d_J(I, I'') = \delta - 1$ and $d_J(I'', I') = 1$, the induction assumption proves the statement. ■

Corollary 3-11: The graph $G_n^r(P)$ is connected.

Proposition 3-12: There exist sequences $I_0, I_1, \dots, I_\lambda \in \mathcal{I}_P^r$ and $a_1, \dots, a_\lambda \in [n]$ such that

$$I_0 \cup I_1 \cup \dots \cup I_i = I_0 \cup \{a_1, \dots, a_i\}, \quad i = 0, 1, \dots, \lambda. \quad (1)$$

Proof: We construct the sequences by induction. By Corollary 2-2 there exists $I_0 \in \mathcal{I}_P^r$.

Assume that for $l < \lambda$ there exist $I_0, I_1, \dots, I_l \in \mathcal{I}_P^r$ and $a_1, \dots, a_l \in [n]$ such that (1) holds for all $i = 0, 1, \dots, l$. We want to find appropriate I_{l+1} and a_{l+1} . Let $N \triangleq I_0 \cup \{a_1, \dots, a_l\}$ and $\mathcal{I}' \triangleq \{I \in \mathcal{I}_P^r \mid I \subseteq N\}$. Since $|N| < |P^r|$, the set $\mathcal{I}_P^r \setminus \mathcal{I}'$ is not empty. By Corollary 3-11 there are two ideals $I \in \mathcal{I}'$ and $I_{l+1} \in \mathcal{I}_P^r \setminus \mathcal{I}'$ such that $d_J(I, I_{l+1}) = 1$. Then I_{l+1} contains exactly one element from $P^r \setminus N$; denote this element by a_{l+1} . So, (1) holds automatically for $i = l + 1$. ■

We use the last proposition to derive the following bound:

Proposition 3-13: $|\mathcal{B}_P^r| \geq 2^{r-1}(2 + \lambda)$.

Proof: Let $I_0, I_1, \dots, I_\lambda \in \mathcal{I}_P^r$ and $a_1, \dots, a_\lambda \in [n]$ be sequences satisfying (1).

Consider the sets $\mathcal{J}_0 \triangleq 2^{I_0}$, $\mathcal{J}_i \triangleq \{\bar{x} \subseteq I_i \mid a_i \in \bar{x}\}$, $i \in [\lambda]$. We have that

- $\mathcal{J}_i \subseteq 2^{I_i}$, $i = 0, 1, \dots, \lambda$;
- the sets $\mathcal{J}_0, \mathcal{J}_1, \dots, \mathcal{J}_\lambda$ are pairwise disjoint;
- $|\mathcal{J}_0| = 2^r$, $|\mathcal{J}_i| = 2^{r-1}$, $i = 1, \dots, \lambda$.

So, $|\mathcal{B}_P^r| \geq |\bigcup_{i=0}^\lambda 2^{I_i}| \geq \sum_{i=0}^\lambda |\mathcal{J}_i| = 2^{r-1}(2 + \lambda)$. ■

Using the ball-packing bound, we deduce the following:

Corollary 3-14: If r -error-correcting $(n, 2^{n-m})$ P -codes exist, then $\lambda \leq 2^{m-r+1} - 2$.

Lemma 3-15: If there exists an r -perfect $(n, 2^{n-m})$ P -code with $r < m$, then

$$m - r < \lambda \leq 2^{m-r+1} - 2.$$

Proof: a) $\lambda \leq 2^{m-r+1} - 2$ holds by Corollary 3-14.

b) By Proposition 2-3 we have $\mathcal{B}_P^r \subseteq 2^{\bar{P}^r}$. If $r < m$, then $|P^r| > r$ and $P^r \notin \mathcal{B}_P^r$. Hence $|\mathcal{B}_P^r| < |2^{\bar{P}^r}| = 2^{r+\lambda}$. Since $|\mathcal{B}_P^r| = 2^m$ by the ball-packing condition, we have $2^m < 2^{r+\lambda}$, i.e., $\lambda > m - r$. ■

IV. $(m-1)$ -PERFECT CODES

Applying Lemma 3-15 for $r = m - 1$ we get the following fact.

Corollary 4-1: (Case $r = m - 1$.) If there exists an r -perfect $(n, 2^{n-m})$ P -code with $r = m - 1$, then $\lambda = 2$.

In the next proposition we describe the structure of a poset P admitting the existence of $(m-1)$ -perfect P -codes. Then, in Proposition 4-3, we prove the existence of $(m-1)$ -perfect P -codes for admissible posets. Theorem 4-4 summarizes the results of this section.

Proposition 4-2: Assume that there exists an r -perfect $(n, 2^{n-m})$ P -code with $r = m - 1$. Then $\mathcal{I}_P^r = \{I \cup \{a_1\}, I \cup \{a_2\}, I \cup \{a_3\}\}$, where $I \in \mathcal{I}_P^{r-1}$ and $a_1, a_2, a_3 \in [n] \setminus I$.

Proof: By Corollary 4-1 we have $\lambda = 2$. By Proposition 3-12 there are $I_1, I_2, I_3 \in \mathcal{I}_P^r$ such that $I_1 \cup I_2 \cup I_3 = P^r$.

By Corollary 3-9 we have that $I_2 \cup I_3 = P^r \setminus \{a_1\}$, $I_3 \cup I_1 = P^r \setminus \{a_2\}$, and $I_1 \cup I_2 = P^r \setminus \{a_3\}$ for some $a_1, a_2, a_3 \in P^r$. This implies that $I_1 = I \cup \{a_1\}$, $I_2 = I \cup \{a_2\}$, and $I_3 = I \cup \{a_3\}$, where $I = I_1 \cap I_2 \cap I_3$. It is easy to see that I is an ideal. ■

By the ball-packing condition we have $|\mathcal{B}_P^r| = 2^m = 2^{r+1}$. Since $|2^{I_1} \cup 2^{I_2} \cup 2^{I_3}| = 2^{r+1}$, there are no other vectors in \mathcal{B}_P^r and there are no other ideals in \mathcal{I}_P^r , i.e., $\mathcal{I}_P^r = \{I_1, I_2, I_3\}$. ■

Proposition 4-3: (Existence of $(m-1)$ -perfect $[n, n-m]$ P -codes.) Let $\mathcal{I}_P^{m-1} = \{I \cup \{a_1\}, I \cup \{a_2\}, I \cup \{a_3\}\}$, where $I \in \mathcal{I}_P^{m-2}$ and $a_1, a_2, a_3 \in [n] \setminus I$. Let $\bar{h}_1, \dots, \bar{h}_n \in F^m$. Assume that \bar{h}_i , $i \in I \cup \{a_1, a_2\}$ are linearly independent and $\bar{h}_{a_3} = \sum_{i \in I} \alpha_i \bar{h}_i + \bar{h}_{a_1} + \bar{h}_{a_2}$ where $\alpha_i \in \{0, 1\}$, $i \in I$. Then the linear code \mathcal{C} defined by $\mathcal{C} \triangleq \{\bar{c} \in F^m \mid \sum_{i \in \bar{c}} \bar{h}_i = \bar{0}\}$ is an $(m-1)$ -perfect P -code.

Proof: The P -code \mathcal{C} is $(m-1)$ -perfect if and only if for each $\bar{v} \in F^m$ there exists a unique $\bar{e} \in \mathcal{B}_P^{m-1}$ such that $\bar{v} + \bar{e} \in \mathcal{C}$, i.e., $\sum_{i \in \bar{v}} \bar{h}_i = \sum_{i \in \bar{e}} \bar{h}_i$. So, it is enough to show that for each $\bar{s} \in F^m$ there exists a unique $\bar{e} \in \mathcal{B}_P^{m-1}$ such that $\sum_{i \in \bar{e}} \bar{h}_i = \bar{s}$.

Since $\{\bar{h}_i\}_{i \in I \cup \{a_1, a_2\}}$ is a basis of F^m , for each $\bar{s} \in F^m$ there exists a (unique) representation

$$\bar{s} = \sum_{i \in I} \beta_i \bar{h}_i + \gamma_1 \bar{h}_{a_1} + \gamma_2 \bar{h}_{a_2}, \quad \beta_i, \gamma_1, \gamma_2 \in \{0, 1\}.$$

Since $\sum_{i \in I} \alpha_i \bar{h}_i + \bar{h}_{a_1} + \bar{h}_{a_2} + \bar{h}_{a_3} = \bar{0}$, we can write

$$\bar{s} = \sum_{i \in I} \beta_i \bar{h}_i + \gamma_1 \bar{h}_{a_1} + \gamma_2 \bar{h}_{a_2} + \gamma_1 \gamma_2 \left(\sum_{i \in I} \alpha_i \bar{h}_i + \bar{h}_{a_1} + \bar{h}_{a_2} + \bar{h}_{a_3} \right).$$

By grouping the terms differently we can rewrite it as follows.

$$\bar{s} = \sum_{i \in I} \beta'_i \bar{h}_i + \gamma'_1 \bar{h}_{a_1} + \gamma'_2 \bar{h}_{a_2} + \gamma'_3 \bar{h}_{a_3},$$

where $\beta'_i \in \{0, 1\}$ and $(\gamma'_1, \gamma'_2, \gamma'_3) \in \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0)\}$. This means that $\bar{s} = \sum_{i \in \bar{e}} \bar{h}_i$ for some $\bar{e} \in \mathcal{B}_P^{m-1}$. Since $|\mathcal{B}_P^{m-1}| = |F^m|$, such a representation is unique. ■

Theorem 4-4: $(m-1)$ -perfect $(n, 2^{n-m})$ P -codes exist if and only if there are $I \in \mathcal{I}_P^{m-2}$ and $a_1, a_2, a_3 \in [n] \setminus I$ such that

- a) if $i, j \in \{1, 2, 3\}$, $i \neq j$, then $\langle a_i, a_j \rangle = \{a_i, a_j\} \cup I$;
- b) for each $a \in [n] \setminus P^r$ there exists $i \in \{1, 2, 3\}$ such that $\{a_i\} \cup I \subseteq \langle a, a \rangle$.

Proof: By Proposition 4-2 and Proposition 4-3 $(m-1)$ -perfect $(n, 2^{n-m})$ P -codes exist if and only if $\mathcal{I}_P^{m-1} = \{I \cup \{a_1\}, I \cup \{a_2\}, I \cup \{a_3\}\}$ for some $I \in \mathcal{I}_P^{m-2}$ and $a_1, a_2, a_3 \in [n] \setminus I$. It is easy to check that this is equivalent to conditions a) and b). ■

V. MORE FACTS

Before dealing with the case $r = m - 2$, it will be useful to prove some more facts. We first show that we can restrict ourselves to consider only the essential part \bar{P}^r of the poset P .

Lemma 5-1: The following statements are equivalent.

- a) There exists an r -perfect P -code \mathcal{C} .
- b) There exists an r -perfect P^r -code \mathcal{C}' .
- c) There exists an $(r-u)$ -perfect \bar{P}^r -code \mathcal{C}'' (recall $u = |P^r \setminus \bar{P}^r|$). The cardinalities of the codes satisfy $|\mathcal{C}''| = |\mathcal{C}'| = 2^{|P \setminus P^r|} |\mathcal{C}|$.

Proof: a) \Leftrightarrow b). By the definition, a perfect code corresponds to a partition of the space into the balls centered in the code vectors. In our case, the ball $\mathcal{B}_P^r = \mathcal{B}_{P^r}^r$ is included in the subspace 2^{P^r} of the space 2^P . Therefore, 2^{P^r} can be partitioned into translations of the balls if and only if 2^P can.

b) \Leftrightarrow c). It is not difficult to see that $\mathcal{B}_{P^r}^{r-u} = \mathcal{B}_{P^r}^r \cap 2^{\bar{P}^r}$ and $\mathcal{B}_{P^r}^r = \mathcal{B}_{P^r}^{r-u} \times 2^{P^r \setminus \bar{P}^r}$. So, if translations of $\mathcal{B}_{P^r}^r$ partition 2^{P^r} , then the intersections with $2^{\bar{P}^r}$ give a partition of $2^{\bar{P}^r}$ into translations of $\mathcal{B}_{P^r}^{r-u}$. And vice versa, having a partition of $2^{\bar{P}^r}$ and multiplying it by $2^{P^r \setminus \bar{P}^r}$ we get a partition of 2^{P^r} .

The relations between the cardinalities immediately follows. ■

Lemma 5-2: If there is an r -error-correcting $(n, 2^{n-m})$ P -code, then the height of \bar{P}^r is not more than $m - r$ (the height is the maximum length of a chain in the poset).

Proof: Assume the contrary, i.e., \tilde{P}^r contains $m-r+1$ pairwise comparable elements $a_0 \preceq a_1 \preceq \dots \preceq a_{m-r}$. Since $a_0 \in \tilde{P}^r$, there exists an ideal $I_1 \in \mathcal{I}_P^r$ such that $a_0 \notin I_1$. Then $a_0, a_1, \dots, a_{m-r} \notin I_1$. Since $a_{m-r} \in \tilde{P}^r$, there exists another ideal $I_2 \in \mathcal{I}_P^r$ such that $a_{m-r} \in I_2$. Then $a_0, a_1, \dots, a_{m-r} \in I_2$. We have that $|I_1 \cup I_2| \geq |I_1 \cup \{a_0, a_1, \dots, a_{m-r}\}| = r + (m-r+1) = m+1 > m$, which contradicts Lemma 3-2. ■

Proposition 5-3: Let U be an upset of P , $l = |P \setminus U| \leq r$, and there be an r -error-correcting $(n, 2^{n-m})$ P -code C . Then

- a) $|\mathcal{B}_U^{r-l}| \leq 2^{m-l}$;
- b) if $|\mathcal{B}_U^{r-l}| = 2^{m-l}$, then C is r -perfect and $\tilde{P}^r \subseteq U$.

Proof: a) It is easy to see that $\mathcal{B}_U^{r-l} \times 2^{P \setminus U} \subseteq \mathcal{B}_P^r$. Since $|\mathcal{B}_P^r| \leq 2^m$ and $|2^{P \setminus U}| = 2^l$, we have $|\mathcal{B}_U^{r-l}| \leq 2^{m-l}$.

b) If $|\mathcal{B}_U^{r-l}| = 2^{m-l}$, then $|\mathcal{B}_P^r| = 2^m$ (i.e., C is r -perfect) and $\mathcal{B}_U^{r-l} \times 2^{P \setminus U} = \mathcal{B}_P^r$. The last equation means that each r -ideal of P is a union of an $(r-l)$ -ideal of U and $P \setminus U$, i.e., $P \setminus U \subseteq \bigcap_{I \in \mathcal{I}_P^r} I$ and, consequently, $\tilde{P}^r \subseteq U$. ■

Recall that k is the number of maximal elements in \tilde{P}^r and $\lambda = |P^r| - r$. (Note that if $|\mathcal{I}_P^r| > 1$, then $\max(P^r) = \max(\tilde{P}^r)$, and thus $k = \max(P^r)$.)

Lemma 5-4: If there is an r -error-correcting $(n, 2^{n-m})$ P -code and $k \geq \lambda$, then

$$2^{r+\lambda} - 2^{r+\lambda-k} \sum_{\sigma=0}^{\lambda-1} \binom{k}{\sigma} \leq 2^m. \quad (2)$$

Proof: Every subset of P^r with not more than $r - (r + \lambda - k)$ elements of $\max(\tilde{P}^r)$ belongs to \mathcal{B}_P^r , because its principal ideal contains the same number of elements of $\max(\tilde{P}^r)$ and at most $|P^r| - |\max(\tilde{P}^r)| = (r + \lambda - k)$ other elements (in total, not more than r). So, the number of such subsets does not exceed $|\mathcal{B}_P^r| \leq 2^m$. On the other hand, this number can be calculated as $|2^{P^r}|$ minus the number $2^{r+\lambda-k} \sum_{j=k-\lambda+1}^k \binom{k}{j} = 2^{r+\lambda-k} \sum_{\sigma=0}^{\lambda-1} \binom{k}{k-\sigma} = 2^{r+\lambda-k} \sum_{\sigma=0}^{\lambda-1} \binom{k}{\sigma}$ of subsets that have more than $k - \lambda$ elements in $\max(\tilde{P}^r)$. ■

Corollary 5-5: If m , r and λ are fixed, then there are only finite number of values of k that admit the existence of an r -error-correcting $(n, 2^{n-m})$ P -code.

Proof: It is easy to see that $2^{r+\lambda} - 2^{r+\lambda-k} \sum_{\sigma=0}^{\lambda-1} \binom{k}{\sigma} \rightarrow 2^{r+\lambda}$ as $k \rightarrow \infty$. By Lemma 3-15 if $r < m$, then $r + \lambda > m$ and Lemma 5-4 proves the statement for $r < m$ (taking into account that only finite number of values of k violate the assumption $k \geq \lambda$).

If $r = m$, then by Theorem 2-6 we have $\tilde{P}^r = \emptyset$ and $k = 0$. ■

Proposition 5-6: Assume that $P = \tilde{P}^r$. Recall that $\lambda = n - r$ in this case. Then

- a) for each $a \in P$ we have $|P \setminus \langle a \rangle| \geq \lambda$;
- b) if there exists an r -perfect $(n, 2^{n-m})$ P -code, then for each $a, a' \in P$ we have $|P \setminus \langle a, a' \rangle| \geq r + \lambda - m$;
- c) for each $b \in P$ we have $|b| \leq \lambda$;

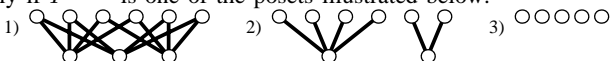
Proof: a) Since $P = \tilde{P}^r$, an element a belongs to at least one r -ideal I . Since $\langle a \rangle \subseteq I$, $|I| = r$, and $|P| = r + \lambda$, there are at least λ elements in $P \setminus \langle a \rangle$.

b) As in p. a), there are r -ideals $I \ni a$ and $I' \ni a'$. Since $\langle a, a' \rangle \subseteq I \cup I'$, the statement follows from Lemma 3-2.

c) Since $P = \tilde{P}^r$, there is at least one r -ideal I that does not contain b . Then the upset $\langle b \rangle$ is disjoint with I and its cardinality does not exceed $|P \setminus I| = \lambda$. ■

VI. THE CASE $r = m - 2$

Theorem 6-1: An $(m-2)$ -perfect $(n, 2^{n-m})$ P -code exists if and only if \tilde{P}^{m-2} is one of the posets illustrated below:



Proof: Assume an r -perfect $(n, 2^{n-m})$ P -code exists with $r = m - 2$. By Lemma 5-1 we can assume $P = \tilde{P}^r$.

By Lemma 5-2 the height of P is 1 or 2. So, P consists of maximal and nonmaximal elements, where each nonmaximal element is also a minimal one. For $a \in P$ denote $\text{valency}(a) \triangleq |\{b \in P \mid b \prec a \text{ or } b \succ a\}|$. Proposition 5-6(a) means that $\text{valency}(a) \leq r - 1$ for each maximal a . Proposition 5-6(c) means that $\text{valency}(b) < \lambda$ for each nonmaximal b .

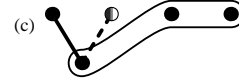
By Lemma 3-15 we have $\lambda \in \{3, 4, 5, 6\}$. So, by Corollary 5-5 the number of admissible values (λ, k) is finite. Note that the case $k \leq 2$ is impossible by Proposition 5-6(b). The other pairs admitting either (2) or $k < \lambda$ are the following: (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5, 5), (5, 6), (6, 3), (6, 4), (6, 5), (6, 6). In all cases we denote by $\{a_1, \dots, a_k\}$ the set of maximal elements of P . Furthermore, we claim that in all cases except (3, 5) the poset contains at least one nonmaximal element. Indeed, otherwise $|P| = k$, $r = k - \lambda$ and, since $m = r + 2$, we have $|\mathcal{B}_P^r| = \sum_{j=0}^{k-\lambda} \binom{k}{j} = 2^{k-\lambda+2}$, which is not true for all considered pairs except (3, 5) (in fact, $P = \max(P)$ means that we have the usual Hamming metric).

Case $\lambda = 3$, $k = 3$. Let $b_1 \in P$ be a nonmaximal element. W.l.o.g. assume $b_1 \prec a_1$. By Proposition 5-6(a) there exists another nonmaximal element b_2 which is noncomparable with a_1 . W.l.o.g. assume $b_2 \prec a_2$. By Proposition 5-6(c) $\text{valency}(b_i) \leq 2$ for $i = 1, 2$. The possible cases are: 1) $b_1 \prec a_1, b_2 \prec a_2$; 2) $b_1 \prec a_1, b_2 \prec \{a_2, a_3\}$; 3) $b_1 \prec \{a_1, a_2\}, b_2 \prec a_2$; 4) $b_1 \prec \{a_1, a_2\}, b_2 \prec \{a_2, a_3\}$; 5) $b_1 \prec \{a_1, a_3\}, b_2 \prec a_2$; 6) $b_1 \prec \{a_1, a_3\}, b_2 \prec \{a_2, a_3\}$. All the cases up to isomorphism are illustrated in the following figures (we emphasize that P can have more elements, but in any case it includes an upset shown in one of the figures, where the dashed lines denotes “optional” relations); Figure (a) corresponds to 1), 3), Figure (b), to 2), 4), 5).



In all these cases we get a contradiction with Corollary 3-4 applied for V being the set of all not shown elements of P and those that are banded by the closed line. The elements of $\succ \max(V) \prec$, which are not in $W(V)$, are marked by black nodes in the figures. We see that $\succ \max(V) \prec$ has at least 4 elements; so, $|W(V)| \leq |P| - 4 < |P| - \lambda = r$, and by Corollary 3-4(b) no r -perfect P -codes exist.

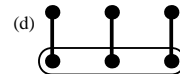
Case $\lambda = 3$, $k = 4$. As proved above, P has a nonmaximal element. Its valency is 1 or 2 by Proposition 5-6(c). The situation is illustrated by the following figure.



As in the previous case we get a contradiction with Corollary 3-4.

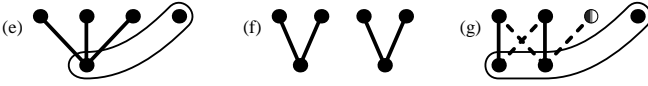
Case $\lambda = 3$, $k = 5$. By proposition 5-3(b) the set of maximal elements coincides with P . In this case the poset metric coincides with the Hamming metric and there exists a 2-perfect repetition code $\{(00000), (11111)\}$.

Case $\lambda = 4$, $k = 3$. By Proposition 5-6(b) for each two maximal elements a, a' we have $|P \setminus \langle a, a' \rangle| \geq r + \lambda - m$. Since $r + \lambda - m = 2$, there is a nonmaximal element b noncomparable with a and a' . So, there is an upset of P illustrated below



and again we get a contradiction with Corollary 3-4.

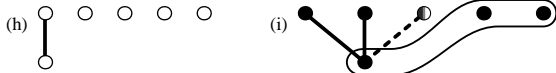
Case $\lambda = 4$, $k = 4$. We claim that there are no subcases different from the ones shown below



As in the previous cases there is a nonmaximal element b_1 and $\text{valency}(b_1) < 4$. Figure (e) illustrates the subcase $\text{valency}(b_1) = 3$. Assume $\text{valency}(b_1) < 3$ and, w.l.o.g., $b_1 < a_1$. By Proposition 5-6(a) the set $P \setminus \langle a_1 \rangle$ contains at least 4 elements, and one of them, say $b_2 < a_2$, is not maximal. Figure (f) illustrates the subcase $b_1 < a_3$, $b_2 < a_4$, or, equivalently, $b_1 < a_4$, $b_2 < a_3$. Figure (g) illustrates the other cases.

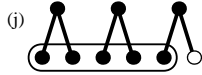
Subcases (e) and (g) contradict Corollary 3-4. In subcase (f) there is at least one more nonmaximal element in P , otherwise $|\mathcal{B}_P| = 12 < 2^m = 16$. All possibilities to add this element lead to subcase (e) or (g).

Case $\lambda = 4$, $k = 5$. There is nonmaximal element and its valency is 1 (Figure (h)), 2 or 3 (Figure (i)).



Subcase (h) contradicts Proposition 5-3(a): $|\mathcal{B}_U^{r-l}| = |\mathcal{B}_U^2| = 18 > 16$ (here and below we take U to be the set of all shown elements; so, l is the number of not shown elements of P). Subcase (i) contradicts Corollary 3-4.

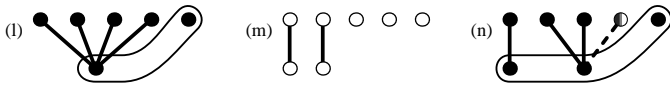
Case $\lambda = 5$, $k = 3$. Similarly to Case $\lambda = 4$, $k = 3$, there are at least two elements of valency 1 under each a_i , $i = 1, 2, 3$. We get a contradiction with Corollary 3-4, see the figure



Case $\lambda = 5$, $k = 4$. Let $b_1 < a_1$. By Proposition 5-6(a) there is nonmaximal element noncomparable with a_1 , say $b_2 < a_2$. By Proposition 5-6(b) there is nonmaximal element noncomparable with a_1 and a_2 . A contradiction with Corollary 3-4, see the figure



Case $\lambda = 5$, $k = 5$. Nonmaximal elements cannot be of valency 4 (Corollary 3-4, see figure (l) below). Assume there is an element $b_1 < a_1$ of valency 1. By Proposition 5-6(a) there is a nonmaximal element noncomparable with a_1 , say b_2 . Figures (m) and (n) illustrate the cases $\text{valency}(b_2) = 1$ and $\text{valency}(b_2) \in \{2, 3\}$.



Case (m) is impossible by Proposition 5-3(a) with $|\mathcal{B}_U^{r-l}| = |\mathcal{B}_U^2| = 20 > 16$. Case (n) contradicts Corollary 3-4.

Let there be a nonmaximal element of valency 2 or 3. By Proposition 5-6(a) there is another nonmaximal element. All the situations up to isomorphism are illustrated by the following figures.



Case (o) contradicts Corollary 3-4. In the case (p) there is one more element in P , otherwise $|\mathcal{B}_P| \leq 14 < 2^m = 16$. But adding a nonmaximal element leads to previous cases.

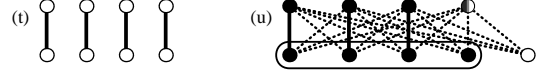
Case $\lambda = 5$, $k = 6$. There is a nonmaximal element b with $\text{valency}(b) \leq 4$. The subcase $\text{valency}(b) = 3$ or 4 (Figure (q)) is impossible by Corollary 3-4; and the subcase $\text{valency}(b) = 1$ or 2 (Figure (r)), by Proposition 5-3(a) with $|\mathcal{B}_U^{r-l}| = |\mathcal{B}_U^2| > 16$.



Case $\lambda = 6$, $k = 3$. Similarly to the Cases (4, 3) and (5, 4), by Proposition 5-6(b) every maximal element covers at least three elements of valency 1. We have a contradiction with Corollary 3-4, see the figure.

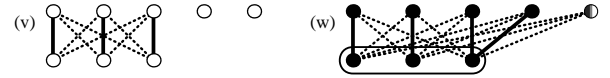


Case $\lambda = 6$, $k = 4$. Let $b_1 < a_1$. By Proposition 5-6(a) there exists a nonmaximal $b_2 \not< a_1$, say $b_2 < a_2$. By Proposition 5-6(b) there exist at least two nonmaximal elements noncomparable with a_1 and a_2 . If there is no more elements in P , then $|P| = 8$ and the valency of each maximal element is 1 by Proposition 5-6(a), see Figure (t) below. The other subcase is shown in Figure (u).



In the subcase (t) we have $|\mathcal{B}_P| = 19$ which is impossible. The subcase (u) contradicts Corollary 3-4.

Case $\lambda = 6$, $k = 5$. Let $b_1 < a_1$. By Proposition 5-6(a) there exists a nonmaximal $b_2 \not< a_1$, say $b_2 < a_2$. By Proposition 5-6(b) there exists at least one nonmaximal element noncomparable with a_1 and a_2 , say $b_3 < a_3$. Let us consider two subcases: (v) b_1, b_2, b_3 are noncomparable with a_4, a_5 ; (w) otherwise:



Subcase (w) contradicts Corollary 3-4. In Subcase (v), if $\text{valency}(a_i) = 1$ for some $i \in \{1, 2, 3\}$, then we have a contradiction with Proposition 5-3(a) ($|\mathcal{B}_U^{r-l}| = |\mathcal{B}_U^2| \geq 18 > 2^{m-l} = 16$). Otherwise $|\mathcal{B}_U^{r-l}| = 16$ and by Proposition 5-3(b) there are no more elements in P . But then there are no r -ideals ($r = 2$) that contain a_1, a_2 , or a_3 , which contradicts our assumption $P = \bar{P}^r$.

Case $\lambda = 6$, $k = 6$. The valency of a nonmaximal element is not more than 5 (Proposition 5-6(c)). Moreover, the valency 5 contradicts Corollary 3-4, see the figure below.



By Proposition 5-6(a) every maximal element is noncomparable with at least one nonmaximal element. Calculate the cardinality of the ball \mathcal{B}_P . The ideal $P \setminus \{a_1, a_2, a_3, a_4, a_5, a_6\}$ gives 2^r vertices. For each a_i there exists an r -ideal containing only one maximal element a_i . This gives $6 \cdot 2^{r-1}$ vertices. So we already have $2^{r+2} = 2^m$ vertices in \mathcal{B}_P and, consequently, there is no other r -ideals. This means that each maximal element is noncomparable with exactly one nonmaximal element. We already proved that the valency of each nonmaximal element is not more than 4, i.e., it is noncomparable with at least two maximal elements. This means that there are 2 or 3 nonmaximal elements. All the cases are illustrated in the following three figures.



It is known [3] that there are perfect P -codes in the cases (y) and (z). We list here examples of such codes.

- (y) linear span($\{1, 2, 6, 7\}, \{1, 3, 4, 5\}, \{2, 3, 8, 9\}, \{1, 4, 6, 9\}$),
- (z) linear span($\{1, 2, 3, 4\}, \{1, 2, 5, 6\}, \{1, 2, 7, 8\}, \{1, 3, 5, 7\}$).

The second code is the Hamming $(8, 2^4, 4)$ code. The first one is a length 9 subcode of the Hamming $(16, 2^{11}, 4)$ code.

It remains to show that there are no 2-perfect P -codes in the case (aa). Assume such a code C exists and, without loss of generality, contains the all-zero vector. By the definition of perfect

P -code $\{2, 3\} = \bar{c}_3 + \bar{b}_3$, $\{2, 4\} = \bar{c}_4 + \bar{b}_4$, and $\{2, 5\} = \bar{c}_5 + \bar{b}_5$, where $\bar{b}_3, \bar{b}_4, \bar{b}_5 \in \mathcal{B}_P^2$ and $\bar{c}_3, \bar{c}_4, \bar{c}_5 \in C$. It is easy to derive from Proposition 3-1 that $\bar{b}_3 \in \{\{4\}, \{1, 4\}, \{5\}, \{1, 5\}\}$ (indeed, otherwise \bar{c}_3 can be covered by two ideals of size 2). So, $\bar{c}_3 \in \{\{2, 3, 4\}, \{1, 2, 3, 4\}, \{2, 3, 5\}, \{1, 2, 3, 5\}\}$; similarly, $\bar{c}_4 \in \{\{2, 3, 4\}, \{1, 2, 3, 4\}, \{2, 4, 5\}, \{1, 2, 4, 5\}\}$, $\bar{c}_5 \in \{\{2, 3, 5\}, \{1, 2, 3, 5\}, \{2, 3, 5\}, \{1, 2, 3, 5\}\}$. In all cases, two vectors from $\{2, 3, 4\}, \{1, 2, 3, 4\}, \{2, 3, 5\}, \{1, 2, 3, 5\}, \{2, 4, 5\}, \{1, 2, 4, 5\}$ belong to C , and we get a contradiction with Proposition 3-1. ■

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